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ON GRAVITATIONAL FIELDS IN RIEMANNIAN SPACES†

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A rigorous foundation will be laid for methods of constructing gravitational fields in given pseudo-Riemannian spaces, particularly Minkowski spaces, with a natural alternative to the global law of universal gravitation, while preserving its local consequences by analogy with local transitions from properties in Euclidean spaces to the properties of Riemannian spaces. There will nevertheless be strong differences in the characteristic relations in global senses.

The essence of the theories developed here is that in topologically equivalent spaces one can use identical fixed coordinate frames of reference with individually defined points. Specially introduced global Lagrangian comoving frames, in space or in mathematically defined model media, defined on appropriate families of time-like world lines L, at each point of which the three-dimensional velocities vanish, are of particular importance. These are coordinate systems in which all individual points of the model spaces or media are at rest, with changes occurring only in global time, which on the world lines of the family is identical with proper time.

Inertial frames of reference—generally local Cartesian tetrads S which, at each point on L, serve as a basis for the introduction of a variety of algebraically and differentially defined mechanical characteristic quantities and, in particular, the four- and three-vectors of absolute velocities and the corresponding absolute accelerations are also of particular importance. These are all fundamental concepts, in terms of which one formulates the basic definitions of mathematical and physical models.

It will be shown that for free gravitational motion of material media, subject only to forces of inertia and body forces—including in particular, gravity—mechanical laws and mechanical phenomena are described in identical terms in comoving coordinates, on the one hand, and in the special frames of reference on the other.

In free flight in space, internal motion within the astronaut's cabin, which takes place under conditions of weightlessness, is therefore described locally in exactly the same way as the analogous mechanical phenomena in inertial systems, where there are no gravitational forces acting on the particles of the medium, as they are cancelled out by forces of inertia. Now this is the situation, in the same sense, both in Newtonian mechanics and in alternative relativistic theories, taking potential energy into account. In general relativity theory (GRT), changes in potential energy, due to changes in the positions of bodies in curved space, are completely eliminated by a suitable choice of a pseudo-Riemannian space.

We shall establish general properties of gravitational fields in pseudo-Riemannian spaces. In particular, we shall show that the density and potential energy of gravitational fields in comoving Lagrangian coordinates along world lines of the family L are constant, though they may differ from one world line of L to another. This is true not only in Newtonian mechanics but also in pseudo-Riemannian spaces.

As we shall see, in relativistic theories it is always necessary to use the law of universal gravitation or some alternative to it due to the additional specification concerning geometrical aspects of pseudo-Riemannian spaces.

It should also be stressed that different choices of the family of world lines L and the presence of point singularities in the field may well cause some solutions of the tensor equation of GRT in a region of empty space to clash with the law of universal gravitation. The same is true of the expressions for the potential energy corresponding to these solutions.

Our main result will be to demonstrate that one can use theoretical solutions of problems in Newtonian

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mechanics in comoving frames of reference (or on the basis of the data of measurements carried out on instruments mounted on moving objects). Computational methods of inertial navigation theory in Riemannian spaces can be used to construct a complete solution of problems involving the determination of the metric and laws of motion of bodies in given pseudo-Riemannian spaces, for arbitrarily given observers.

The agreement between GRT and Newton's theory (in the basic approximation) is due to the fact that over short time intervals planetary orbits are almost straight lines in Euclidean space or Schwarzschild geodesics.

1. THE CONSTRUCTION of models for gravitational fields will rely on the fundamental variational equation [1–4]

$$\delta \int_{V_4} \Lambda dV_4 + \delta W^* + \delta W = 0$$

which expresses the universal thermodynamic principles of physics and encompasses all possible hypotheses concerning interactions in model media, based on experiment or observation, for various kinds of natural or artificially created or projected phenomena.

The fundamental equation, considered in appropriately specialized formulations for special cases, as applied in analytical mechanics to obtain three-dimensional equations, is in complete agreement with a variety of logically postulated and accepted "variational principles". At the same time, it enables one to formulate not only the three-dimensional Euler equations, but also to derive equations of state for internal interactions in various kinds of models of material bodies or fields.

Thus, the basic equation provides a point of departure for deriving a closed system, including all necessary equations and characteristic conditions (initial data, boundary conditions, etc.). The principles of model construction are related to the postulation of Lagrangians ΛdV_4 and appropriate terms in δW^*, representing the total local energy of the model, expressed in terms of a system of characteristic defining parameters.

In accordance with the main, well-tried scientific ideas of model construction in a variety of physical fields (though not all), one can postulate that the total elementary energy ΛdV_4 and global energy $\int_{V_4} \Lambda dV_4$ can be expressed as the sum of different forms of energy, whether of the same or different natures, which may be converted into one another in the phenomena under study.

Further constructions are based on several ideal notions:

1. a four-dimensional pseudo-Riemannian space, defined when investigating GRT, or prescribed in advance when the problem is set up, such as Euclidean space in Newtonian mechanics or, for example, Minkowski space in special relativity theory (SRT) or certain other Riemannian spaces, including Schwarzschild spaces, etc.;

2. a moving continuous medium, as a physical model with individual elements, say, with intrinsic masses dm = const; possible generalizations in which dm is allowed to vary, though not out of the question, will not be touched upon in the theory developed below;

3. a gravitational field, generated by a mass distribution with density ρ or—for fields corresponding to media—comprised of elementary objects distributed in a continuum, with $dm = 0$ but possessing momenta and energy characteristics for gravitational fields generated by charges and magnets.

Each of these positions involves different assumptions, some of which are already universally accepted in macroscopic scientific theories, while others are new.

Below we will consider non-linear relativistic theories of gravitational fields and laws of motion of material points as elements of a certain family of world lines L in Riemannian spaces, where the latter are either to be determined, or prescribed in advance as carriers of gravitational fields generated by distributed masses, with distributed energy of test particles for the scalar characteristics of the field, and, in particular, with universal fixed Riemannian spaces, selected in advance, as is assumed for Euclidean space in Newtonian mechanics or, as we shall show, in SRT for Minkowski space or in GRT after the approximate solutions have been replaced by exact solutions in the formulations of the problems (for example, the use of a fixed Schwarzschild space in the theory of planetary motion around the Sun).

We will now outline the basic definitions of the characteristic parameters in the expressions that represent different forms of energy in volumes of the Riemannian spaces; in particular, we shall introduce the concept of gravitational energy.

Introducing volume integrals $\delta\int_{V_4}\Lambda dV_4$ for the total expressions of energy variation in four-spaces, we let V_4 denote an arbitrarily chosen substantional invariantly defined four-dimensional volume in Riemannian space. Accordingly, we have an invariantly defined four-dimensional element $dV_4 = dV_3 \cdot d\tau$. For the continuous medium under consideration, dV_3 denotes an invariantly defined infinitely three-dimensional volume element (a particle) and $d\tau$ a time-like element in the non-holonomically distributed volume dV_3, and τ denoting proper time, defined globally in the relevant system of preassigned or unknown world lines L for individual points, with equations $\xi^\alpha = \text{const}$, $(\alpha = 1, 2, 3)$, for which we introduce, without loss of generality, comoving Lagrangian coordinates $(\xi^1, \xi^2, \xi^3, \tau)$ and canonical metrics in the form

$$ds^2 = c^2 d\tau^2 + 2g_{\alpha 4}(\xi^\gamma, \tau) d\xi^\alpha d\tau + g_{\alpha\beta}(\xi^\gamma, \tau) d\xi^\alpha d\xi^\beta; \quad \alpha, \beta = 1, 2, 3$$

(It should be noted that one cannot in general use the global equality $V_4 = \tau V_3$ with finite or infinite volumes V_4 and V_3 for finite or infinitesimal values of τ.)

We will now give a few definitions.

As frames of reference we shall use local inertial tetrads S at each point, on L we have $ds^2 = c^2 d\bar{\tau}^2 = l^2 d\bar{\tau}^2$, where $l = qc$ is a long distance, q is some characteristic time constant, and $\bar{\tau}$ is a dimensionless time coordinate. At the points of the world lines L we have the following formula for the four-dimensional velocity: $\mathbf{u} = \mathbf{ds}/d\tau = \bar{c}$ with $|c| = \text{const}$; the absolute acceleration is $\mathbf{a} = d\mathbf{u}/d\tau$. We will define the concept of mass density by $\rho = dm/dV_3$, where dm is the element of rest mass for an individualized volume dV_3 in the local inertial tetrads along the world lines L.

Fundamental physical laws are established and formulated in terms of local inertial tetrads S with constant bases e^i, as frames of reference with locally defined coordinates x^α along world lines L and global time τ in the canonically defined comoving metrics indicated above; one also uses the concepts of velocity and components of absolute acceleration, for which the following formula holds

$$a_\alpha = c\,\partial g_{\alpha 4}(\xi^\gamma, \tau)/\partial\tau, \quad a_4 = 0; \quad \alpha, \gamma = 1, 2, 3.$$

Note that in comoving coordinates along an arbitrary but fixed line L with $\xi^\alpha = \text{const}$, the coordinate τ is proper time.

In any coordinate frame x^1, x^2, x^3, τ', the equation of each individual line L^* in the specified space may be written as a coordinate transformation

$$\begin{aligned} &x^\alpha = f^\alpha(\xi^1, \xi^2, \xi^3, \tau) \quad \text{or} \quad \xi^\alpha = \xi^\alpha(x^1, x^2, x^3, \tau) \\ &\text{and} \quad \tau' = \tau; \quad \alpha = 1, 2, 3 \end{aligned} \tag{1.1}$$

which preserves the invariance of the comoving metric of the pseudo-Riemannian space, generally speaking, only in an infinitesimal neighbourhood along L; however, the totality of all transformations of this kind on different world lines L, with difference functions f^α, will not generally form a unified frame of reference for the same Riemannian space.

We will now consider a certain line L^* in isolation; suppose that at each of its points we have a system of variable bases ε_i, defined in terms of the components of the metric tensor $g_{ij}(\xi^\alpha, \tau) = (\varepsilon_i \varepsilon_j)$ and the Christoffel symbols

$$\Gamma^i_{pq} = \frac{1}{2} g^{is} \left(\frac{\partial g_{ps}}{\partial \xi^q} + \frac{\partial g_{qs}}{\partial \xi^p} - \frac{\partial g_{pq}}{\partial \xi^s} \right)$$

These bases satisfy the following formulae

$$\frac{\partial \varepsilon_i}{\partial \xi^k} = \Gamma^s_{ik}(\xi^\gamma)\varepsilon_s, \quad \text{or} \quad \frac{\partial \varepsilon_i}{\partial x^k} = \Gamma^s_{ik}(x^\gamma)\varepsilon_s$$

$$s, i, k, \gamma = 1, 2, 3, 4$$

On the world line L^*, apart from the basis ε_i, we also introduce inertial tetrad bases e_i, such that

$e_i = \varepsilon_i$ or $e_i \neq \varepsilon_i$ at the point under consideration; however, as functions of x^i and ξ^i these bases are constant at each point of L^*

$$\frac{\partial e_i}{\partial \xi^k} = \frac{\partial e_i}{\partial x^k} = 0; \quad i, k = 1, 2, 3, 4$$

All the Christoffel symbols vanish for such inertial bases e_i. Obviously, using the transformation laws for Christoffel symbols along L^*, we thus obtain the following formulae for the functions $x^\alpha(\xi^\gamma, \tau)$

$$\frac{\partial^2 x^k}{\partial \xi^i \partial \xi^j} = -\Gamma^k_{pq}(\xi^\gamma, \tau) \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \tag{1.2}$$

These formulae, considered as equations, essentially determine the functions (1.1) only along world lines L^*. Moreover, they are not integrable in finite volumes of the Riemannian space, since it is usually not possible to introduce Cartesian coordinates in Riemannian spaces, with the exception of Minkowski spaces.

In Newtonian mechanics and SRT one can in fact define a transformation $x^\alpha = x^\alpha(\xi^\alpha, \tau)$ and $\tau' = \tau$ relative to which Eqs (1.2) hold globally. In Riemannian spaces such a global transformation may also be introduced, but then it may happen that Eqs (1.2) are satisfied only on an isolated, though arbitrarily specified, world line L.

For each arbitrary line L^*, a transformation (1.1) also exists in the comoving coordinate frame known as the Fermi frame; in variables x^i on L^* we have $\Gamma^k_{ij}(x^\gamma, \tau) = 0$, consequently, $\partial g_{ij}(x^\alpha, \tau)/\partial x^s = 0$.

In the coordinate frame x^i we may assume that, to within higher orders, the elements dV_3 in tetrads S for different points of L^* are identical in the limit.

Indeed, it follows from the equation of continuity in the Fermi frame x^i along any world line L^* that

$$\frac{\partial \rho u^1}{\partial x^1} + \frac{\partial \rho u^2}{\partial x^2} + \frac{\partial \rho u^3}{\partial x^3} + \frac{\partial \rho u^4}{\partial x^4} = 0$$

In the comoving coordinate frame, when $u^4 = 1$ and $u^\alpha = 0$ on L^*, we obtain: $\partial \rho/\partial \tau = 0$ both in variables x^i and in the variables ξ^i with bases $e_i = \text{const}$.

In addition, it follows from the equation of continuity that

$$\rho dV_3 = dm \tag{1.3}$$

identically along world lines L with different dm and ρ but with kinematically identical dV_3 at all points of the Riemannian space for identically introduced tetrads S.

It follows from (1.3) that the invariants $\rho_{(e')}$, dV_3 and dm thus defined on comoving lines L depend only on the three arguments x^1, x^2 and x^3 or on ξ^1, ξ^2, ξ^3 along the line L, but not on proper time τ along L.

On the other hand, from the point of view of the reference frame of an arbitrarily specific observer in the same space, with other coordinates $\eta^1, \eta^2, \eta^3, \tau' \neq \tau$, it is essential that these invariants depend on all four variables.

In the comoving reference frame, individual mass particles are in a state of relative rest, in which only time is changing; relative to the inertial reference frames for individual points, however, the motion of the particles is generally accelerated. On the other hand, in free motion of the particles, under the sole influence of body forces of gravity, these forces are completely cancelled out by the forces of inertia; particles at rest relative to the comoving reference frame are in a state of weightlessness, similar to the rest state of particles relative to the Fermi local inertial reference frames. All this explains why it is necessary to use canonical comoving coordinate frames, particularly in the theory of gravitation, which involves only body forces of gravity and forces of inertia.

If the problem in question involves the determination of quantities of various types in preassigned reference frames in globally defined fixed spaces, then transformations of the solutions from the comoving reference frames to the preassigned frames result in a problem in the theory of inertial navigation.

We have published a solution to such navigation problems for Riemannian spaces. Solutions expressed in Lagrangian coordinates may be carried over to an arbitrary prescribed metric as a reference frame in a given Riemannian space in GRT, using the algorithms of inertial navigation theory published in 1976 [5]. Different versions of inertial navigation theory for specific bodies in a Newtonian framework were first worked out in Tkachev's doctoral dissertation in 1944 (see also [6]).

In experiments or in automatic devices, it is natural to mount measuring instruments on moving bodies and thus to obtain measured values of the quantities that characterize moving individualized objects in Lagrangian variables.

We must add that the fundamental physical laws derived or verified in experiments are usually formulated for individually defined model objects.

2. We will now list the elements of variable energy in our models, for a Riemannian space and for a material medium with moving elements individualized by means of the coordinates ξ^1, ξ^2, ξ^3, τ of particles of the medium with masses dm.

In accordance with our fundamental assumption, we define the elementary energy in a volume integral with respect to a substantial volume V_4 as the sum of the different forms of energy in the elements dV_4 of the finite volume V_4. We will be governed in this respect by various scientific ideas from applied theories that have been found to agree with experiment in the mechanical contexts considered here.

Experimentally confirmed and proven theses dictate that we use four-dimensional pseudo-Riemannian spaces with signatures "$---+$" and take the expression

$$\frac{R}{2k}\,dV_4 \tag{2.1}$$

as the variable energy density of a geometrical volume element dV_4 due to the Gaussian curvature R of the Riemannian space. (A similar definition of energy, maintaining its scalar nature and linking it to the curvature of continua, whether as a function of the curvature itself or generated by variations of curvature, may be found in elasticity theory.)

The coefficient $k = 8\pi G/c^4 = 2.1\times 10^{-48}$ sec^2/g cm is the "gravitational constant"; G is the empirical constant in Newton's law of universal gravitation. The small factor $2\pi/c^4$ arises from the requirement that GRT and Newtonian theory agree in the limit for free motion of individualized material points in a vacuum over short intervals of time—where GRT assumes the motion to take place along geodesics, and Newtonian mechanics views it as accelerated motion for small τ and small three-dimensional velocities.

It is obvious that if $R/(2k)$ is finite, then the Gaussian curvature R and the important combination $R^{ij} - \frac{1}{2}g^{ij}R$ in the field equation (2.10) (see below) are very small functions of points in the Riemannian spaces; in GRT, in four-dimensional volumes of empty Riemannian spaces, the "gravitational constant" k is assumed to have no effect on the equation of motion of material particles in a vacuum. Nevertheless, the influence of k still manifests itself in the characteristics of singular points and in the boundary conditions.

In generalized models of GRT, k may be defined as a variable physical parameter; this was indeed proposed by Dirac in his lecture of August 1975 in Australia [3].

In a moving matter model, the scalar mechanical thermodynamic energy, which depends on the masses of the particles in the medium, is defined as the sum of the energy elements of the masses in the medium, given by the formula

$$\rho g_{ij} u^i u^j dV_3 d\tau = dmc^2 d\tau \tag{2.2}$$

which, in particular, generates a body force of inertia in the Euler equations with dm = const.

It is taken for granted here that the energy of a particle, as such, is independent of the position of the particle in space. Experiments in Newtonian physics, however, have shown that in the presence of a gravitational field one may—and should—also introduce the potential energy of material points in a body as a function of their positions.

A unique feature of GRT is that it does not assign a potential energy, dependent on position, to masses; instead, it assumes certain effects due to the curvature of space. Newtonian mechanics takes potential energy into consideration explicitly, which is quite natural. This replacement of the potential energy of a position by geometrical properties of four-dimensional space greatly complicates the mathematical and physical aspects of the possible limiting transition from GRT to authoritative Newtonian mechanics.

At the same time, these complications in GRT generally make it impossible to proceed in the limit from GRT to Newtonian mechanics over long intervals of time.

The following considerations are basic. In a fixed Riemannian space there are various possible motions of material media and various fields, including gravitational fields, generated only by the distribution of moving masses and their velocities or of massless particles possessing energies and momenta, or by the corresponding elements of the media, which possess charges and magnetization when electromagnetic forces act on the test particles.

The potential energy of the fictitious state of rest of the elements of a continuous mass distribution in volumes V_4 of a gravitational field is defined by the formula

$$\rho \frac{dU}{d\tau} dV_4 = dm \frac{dU}{d\tau} d\tau = dmdU \tag{2.3}$$

where the scalar function of the coordinates $U(\xi^i) = U(x^i)$ is the specific potential energy per unit test mass induced by the field; this energy is assigned to the elements of the material medium as a function of their positions in space.

It is of great importance that the total potential energy of the points of a gravitational field, as a physical characteristic of the field for individual points of the space, is proportional to the masses of the test particles, which may differ for a fixed function U of the coordinates. We stress that in analytical mechanics of systems with a finite number of degrees of freedom the potential energy always occurs in the equations and is directly involved in the Hamilton variational principle. Under general conditions, the mechanical characteristics of moving systems depend on the geometry of the space; at the same time, one postulates that the potential energy of the gravitational field or the Gaussian curvature of the Riemannian space, which occur in the energy formula (2.8) (see below) in specified spaces, may be treated, respectively, as due both to the curvature of the Riemannian space and to the presence of a gravitational field with the variables generated by the body forces of gravity, which are defined in general cases in terms of the scalar functions $U(\xi^\alpha, \tau) = U(z^\alpha, \tau)$.

In GRT, energy effects are generated by the curvature R in the absence of the term (2.3), but Newtonian physics attributes them solely to the acceleration of the gravitational force, $\mathbf{g} = -\operatorname{grad} U$, in a two-dimensional Euclidean space. Our formula (2.3), with the total derivative $dU/d\tau$ in place of simply U, is motivated by the model requirement that the specific potential energy U be single-valued as a function of the points of the Riemannian space.

We emphasize that the scalar field U, which has the dimension of velocity squared, appears only when there are particles (elements of the medium) present in the field that possess masses or momenta in the form of their energies and body forces of gravity.

The specific gravitational energy per unit mass, for an infinitesimal particle moving in space and in time, occurs through the differential $dU = U(P') - U(P)$, where P' and P are adjoining positions occupied consecutively by distinguished elements of the medium. In the continuum theory, this formula, as written, holds for the motion of a particle along its world line L in any coordinates x^i and formula (2.3) may be written in the form

$$\rho \frac{dU}{d\tau} dV_4 = dm \frac{dU}{d\tau} d\tau = dm \left(\frac{\partial U}{\partial x^1} dx^1 + \frac{\partial U}{\partial x^2} dx^2 + \frac{\partial U}{\partial x^3} dx^3 + \frac{\partial U}{\partial \tau} d\tau \right) \tag{2.4}$$

When dealing with gravitational fields in various models of moving media or variable fields, one must also introduce, besides the energy elements (2.1)–(2.4), an analogous form of energy, U^* generated in the general case by internal and external surface forces of interaction. This energy is measured per unit mass dm or unit three-dimensional volume dV_3 of the moving medium of the field; it is represented in Λ by terms of the form

$$dV_3 \frac{dU^*}{d\tau} (\mu^k, \nabla_i \mu^k, \dots) d\tau \tag{2.5}$$

The function U^* depends on additional physical and mechanical parameters μ^k of a scalar and tensor nature, on given or unknown characteristics. In particular, one may also require the imposition of various constraints, which accordingly introduce added terms in the Lagrangian with Lagrange multipliers λ^p.

The scalar energy component (2.5) represents the densities of different kinds of physical energy of a thermodynamical nature or special external constraints in internal processes in elements dV_4 for the volume of media, fields and world lines under consideration, due to the introduction of additional mathematically motivated constants or variable parameters μ^k or combinations of such parameters with mechanical parameters and their derivatives with respect to the coordinates and the time τ. (For example: temperature, the components of the strain tensor and moment characteristics in elasticity theory, electromagnetic effects in radiation or chemical and nuclear reactions, etc.)

In relativistic theories, as in Newtonian physics, terms involving $U \neq$ const and U^* lead to families of world lines of motion of individual points L with absolute accelerations that may be non-zero.

Besides the energy components (2.2), (2.4) and (2.5), the volume integral for the total energy will also include an invariantly defined divergent term

$$\frac{c^2}{4\pi\gamma}\nabla^i\nabla_i U dV_4 = -\frac{c^2}{4\pi\gamma}(\text{div grad } U)dV_4 \tag{2.6}$$

where c^2 and γ are dimensional constants; γ is introduced in order to assure that the various terms in the sums representing the Lagrangian will have the same dimensions.

In view of the dimensional relations $[k] = T^2/(ML)$ and $[\gamma] = L^3/(MT^2)$ and the conclusions of the theory developed below, the agreement between Newtonian mechanics and experiment implies that one can take $\gamma = kc^4/(8\pi) = G$, where G is the gravitational constant in the universal law of gravitation.

It is obvious that the divergent term (2.6) has no effect on the Euler equations for the variations δg_{ij} and δx^i.

The local energy (2.6) arises from the work of external gravitational forces on the boundaries of the finite volume V_4 which occurs in the basic equation in the term δW now being defined, which is a surface integral over the boundary Σ of V_4 (to be precise, a three-dimensional surface integral, once the divergent term in the volume integral has been transformed to an integral over the boundary of V_4). This gives the following expression for the surface force per surface element $d\Sigma'$ due to the component (2.6)

$$\frac{c^2}{4\pi\gamma}\text{grad } U n d\Sigma \tag{2.7}$$

Each of the constituents described above involves assumptions, some of which are already universally accepted in scientific macroscopic versions of GRT, but the terms (2.4), (2.5) and the terms of the form $(dU^*/d\tau)dV_4$ and (2.6) in the expression for ΛdV_4 in a Riemannian space have been added here.

In classical theories, in the case of reversible conservative models with gravitational fields and Riemannian spaces, it is taken for granted that $\delta W^* = 0$, and if $U^* \neq 0$ one can limit consideration of the volume integral in the basic equation to the following expressions

$$\delta \int_{V_4} \Lambda dV_4 = \delta \int_{V_4} \left[-\frac{R}{2k}dV_4 + \rho g_{ij}u^i u^j dV_4 + dm dU + \frac{c^2}{4\pi\gamma}\nabla^i\nabla_i U dV_4 + \frac{dU^*}{d\tau}dV_4\right] \tag{2.8}$$

For individual particles, the condition $dm(\xi^\gamma) =$ const may be viewed as a constraint, corresponding to the hypothesis of mass conservation for different variable three-dimensional volume elements—the particles in the models of gravitational fields that we are constructing. The variation symbol δ represents only the replacement of real increments by conceptual, virtual increments, with allowance for the constraints dictated by the formulation of the model theories.

Writing formula (2.8) in an invariant form in the comoving coordinate frame, one can evaluate the infinitesimal variations by subjecting the space itself to geometrical variations and considering the properties of the relevant world lines L. When the variation is applied, one should consider, besides the comoving reference frame in variables ξ^α and τ, also the locally defined inertial reference frame of the observer with transformed variables x^α, τ

$$x^\alpha = x^\gamma(\xi^1, \xi^2, \xi^3, \tau); \qquad \delta x^\alpha = \frac{\partial x^\alpha}{\partial \xi^\gamma}\delta\xi^\gamma + \frac{\partial x^\alpha}{\partial \tau}\delta\tau \quad \text{and} \quad \tau' = \tau + \delta\tau \tag{2.9}$$

The variations δx^α, $\delta\tau$ and $\delta\xi^\alpha$, $\delta\tau$ represent arbitrary virtual (conceptual) infinitesimal deviations of the quantities in question from their (unknown) real values, where the latter are to be obtained by solving the mechanical problems under consideration.

It should be pointed out explicitly that in every fixed Riemannian space, whether given or to be determined, U may be a different function of the coordinates, depending on the problem being treated.

The determination of U as a function of the mass distribution ρ is governed by additional assumptions, all arising from the law of universal gravitation. Accordingly, all other things being equal, gravitational fields may take different forms in the same Riemannian space, just as in Euclidean space in Newtonian mechanics.

The deviation of the volume Euler equations as coefficients of arbitrary variations δg_{ij} yields the field equations; variations δx^α, $\delta\tau$, however, lead to the equations for the world lines L, which are actually local laws of conservation of momentum. Clearly, the last term but one in the integral (2.8), which is of divergent form, transforms into a surface integral for δW over Σ and therefore does not influence the Euler equations for the field or the equations of motion. However, the solution of specific problems relating to the equation of state, which appears in the boundary and initial conditions, is also affected by the divergent terms in the expression for Λ.

If the condition $U^* = \text{const}$ is not assumed, the volume field equations, obtained as coefficients of the variations δg_{ij}, are determined only by the first two terms in formula (2.8) and, as is well known and indeed obvious, these equations may be written in the form

$$R^{ij} - \tfrac{1}{2}g^{ij}R = k\rho u^i u^j \tag{2.10}$$

In empty spaces one has $\rho = 0$, implying the following equations for volumes in Riemannian space geometry which are not directly distorted by the presence of various objects and events

$$R^{ij} - \tfrac{1}{2}g^{ij}R = 0 \tag{2.11}$$

and consequently $R = 0$ and $R_{ij} = 0$.

In that case the Riemann tensor is equal to the Weyl tensor, and Eqs (2.11) have a large set of different solutions.

The solutions of the field equations (2.11) are adapted to specific situations by imposing additional conditions, under which these equations have unique solutions. Such conditions may reduce to the specification of families of world lines L and of singularities at isolated points for types of solutions with matrices K in the sense of Petrov. They are determined using canonical three-dimensional symmetric matrices M and N in the orthonormal tetrads S, expressed in terms of the roots $\lambda_s = -(\alpha_s \pm i\beta_s)$ of the "secular equation" of the six-dimensional matrix K, whose entries are the components of the Weyl tensor.†

The possible distributions of these roots as functions of the points of the space are established using the Bianchi identities. One constructs solutions of the problems that arise with families of world lines L that correspond to the metric of the Weyl tensor in the comoving systems of Lagrange coordinates, in the fundamental spaces determined from Eqs (2.11) as functions of the accelerations on world lines L. These in turn are determined as functions of the distributed density of matter ρ, whether prescribed or to be determined, in other regions in the space determined by the solution of Eq. (2.10).

In the problem thus stated, relying on the basic equation taking into account the thermodynamic energy U^* per unit volume dV_3, we can write for variations δx^α, $\delta\tau$

$$\delta\left[\left(\rho g_{ij}u^i u^j + \rho\frac{dU}{d\tau}\right)dV_3 d\tau + \frac{dU^*}{d\tau}dV_4\right] = 0 \tag{2.12}$$

†See SEDOV, L. I., On the properties of invariant components of the Riemann tensor for $T_{ij} = \kappa g_{ij}$ that follow from the Bianchi identities and the equalities $\alpha_1 + \alpha_2 + \alpha_3 = \kappa$ and $\beta_1 + \beta_2 + \beta_3 = 0$. Preprint, Institute of Mechanics, Moscow State University, 1992.

The differentials $d\tau$ occurring in dV_4 and in $dU/d\tau$ may be replaced, essentially without after-effects, by $d\tau/q$, where q is any scaling constant with the dimension of time. The quotient $d\tau/q = d\bar{\tau}'$ may be introduced directly as an invariant global abstract time coordinate on the family of world lines L. Consequently, the terms $dmc^2 d\bar{\tau}'$ and $dm(\partial U/\partial\bar{\tau}')d\bar{\tau}'$ have the dimensions of energy, in keeping with the physical sense of the basic variational equation.

The scalar relationship (2.12) can be written in more detail in the comoving reference frame, in which the world lines correspond to the global proper time coordinate τ. In that connection we point out here the following starting formulae in the variables ξ^α and τ, which also hold in the variables $x^\alpha(\xi^\gamma, \tau)$ and τ (for $c = 1$) on the world lines L

$$u^4 = 1, \quad u^1 = u^2 = u^3 = 0; \quad u_4 = 1, \quad u_\alpha = u^k g_{k\alpha} = g_{4\alpha}(\xi^\gamma, \tau)$$

where, since $g_{ij}u^i u^j = u_j u^j$, before varying x^α and τ along L, we can write

$$\frac{d}{d\tau}(u_j u^j)\,d\tau = \frac{du_j}{d\tau}dx^j + u_j\frac{du^j}{d\tau}d\tau = \frac{du_\alpha}{d\tau}dx^\alpha$$

$$\rho dV_3 = dm = \text{const}$$

because in the canonical comoving coordinates, everywhere and always, we have $u^4 = 1$, $u^\alpha = 0$, $u_4 = 1$, and

$$\frac{\partial u_\alpha}{\partial\tau} = \frac{\partial g_{\alpha 4}(\xi^\gamma, \tau)}{\partial\tau} = a_\alpha, \quad a_4 = 0 \tag{2.13}$$

where a_α are the components of the absolute acceleration at the points of the coordinate time world line L.

Now, using (2.12) with $U^* = 0$ (only gravitation present), the variation of (2.12) for a small test mass becomes

$$dm(a_\alpha \delta x^\alpha + \delta U) = 0 \quad \text{or}$$

$$dm\left(a_\alpha\delta x^\alpha + \frac{\partial U}{\partial x^\alpha}\delta x^\alpha + \frac{\partial U}{\partial\tau}\delta\tau\right) = 0$$

(The expression for the varied first term when dm = const, which is $dma_\alpha\delta x^\alpha$, is quite well known.) Hence one immediately obtains the fundamental equations

$$a_\alpha = g_\alpha = -\partial U/\partial x^\alpha \text{ and } \partial U/\partial\tau = 0 \tag{2.14}$$

Written differently, in three-dimensional vector notation, we have $\mathbf{a} = \mathbf{g}$ and $\mathbf{g} = -\text{grad}U$ when $\partial U/\partial\tau = 0$ or $U(x^1, x^2, x^3)$, and the derivative of the potential energy U per unit mass on world lines L (orbits) must vanish: $\partial U/\partial\tau = 0$. We have thus shown that the characteristic of the potential energy of dm particles-elements of the medium introduced by Eq. (2.3) depends only on the variables x^1, x^2, x^3.

If $x^\alpha = \xi^\alpha$, then, in the comoving coordinate frame, in which ξ^α = const on world lines L, we obtain $U = U(\xi^1, \xi^2, \xi^3)$ (the global equalities $U(x^1, x^2, x^3) = U(\xi^1, \xi^2, \xi^3)$ arise from the use of inertial tetrads S, which are constant at each point of the world lines L). Hence it follows that in regions of empty space at different points of the same world line L, U has a constant value but, as a rule, will take different values on different Ls if $\partial U/\partial\xi^\alpha \neq 0$ for $a_\alpha = 0$.

In that case, Eq (2.14) shows there must be a gravitational field of accelerations, and therefore the orbits L in the Riemannian spaces will not be geodesics.

The situation we have just described seems at first sight to be paradoxical. However (e.g. in particular, in Newtonian mechanics), it is perfectly clear that in the comoving coordinate frame ξ^α, τ of Earth the field of gravitational accelerations generated by the Earth is stationary, but it is not stationary in the Copernican frame y^α, τ' associated with the Sun.

This example of describing the motion of a planet in the coordinate frame y^α, τ' indicates the existence of possible gravitational waves, perturbed by moving masses, which propagate at arbitrary velocities less than c.

The above conclusions are quite obvious if the Earth is treated as an absolutely rigid body, but their validity for deformable systems, when the components of the metric in the comoving metrics depend on the time τ, stems from the fact that individual points in the corresponding reference frames maintain a state of rest, whose main mechanical characteristics, according to the formulation of the problem, are the constant comoving coordinates $\xi^\alpha = \text{const}$.

The model concepts and the resultant constructive theories for the systematic and effective description of events in natural science and technology, as developed in Newtonian mechanics, are tremendous scientific achievements. In many cases they attain an extraordinary degree of accuracy, sufficient to represent reality in many situations; at the same time, they furnish a solid base for further generalizations of the fundamental model concepts in relativistic theories.

Naturally, further improved versions of the scientific theories, aimed at accounting for experimental effects that contradict Newtonian mechanics, may be created in the first instance by modifying one's concepts of space and time. For example, one can introduce a four-dimensional pseudo-Riemannian space, which basically retains the cardinal ideas of the field of gravitation and its characteristic features.

3. According to Newton, $U \neq 0$ in Euclidean space, but in GRT, as a rule, the scalar curvature R or other invariants of the Riemann tensor may be non-zero. However, any limiting procedure leading from GRT to Newtonian mechanics must result in $R = 0$ and $R_{ijkl} = 0$.

In the previous sections we outlined a relativistic theory in which, on passing to Newtonian mechanics, one obtains $U \neq 0$, so that near the limits $U \neq R/(2k)$; in particular, in Schwarzschild solutions $R = 0$.

In the comoving Lagrange coordinates, there is for every individualized point (particle) a corresponding rest state, while the assigned masses dm and elements of total specific energy U_0 are constant on L. (In rocket flight, when allowance is made for various radiations, nuclear reactions, etc., further complications may be necessary in the theory, which must adopt appropriate hypotheses or laws for the variable quantities dm and U_0 [7].)

In general cases, however, in canonical comoving coordinates for Riemannian spaces, on passing from a line L to an adjoining line L' one has

$$U' - U \neq 0 \text{ and } \partial U/\partial \xi^\alpha \neq 0 \tag{3.1}$$

and, by Eq. (2.14), the absolute acceleration $\mathbf{g}$ relative to the inertial tetrads will satisfy the formula $\mathbf{g} = -\text{grad}U$.

Proceeding as in Newtonian mechanics, one can also obtain a momentum equation for celestial mechanics.

Since the world lines L in GRT are geodesics, it follows that $\mathbf{g} = 0$ in GRT, and therefore U_0 is an absolute constant in volumes of empty space.

On the other hand, we know that in GRT the integral (2.8) does not contain terms with $\rho(dU/d\tau)\,dV_4$, which is the same as saying that U has a constant value at all points of a volume of the Riemannian space. As GRT does not allow for interactions among the planets, this fact imposes a significant restriction on the form of the possible orbits L in the Schwarzschild field.

In Riemannian spaces, when the energy-momentum tensor is non-zero, the specific energies U and U^* in (2.3) and (2.7) are generally also non-zero, while the relevant world lines in the family L in the canonical comoving metric correspond to accelerated motions of the individual points of the moving medium.

Thus, in general situations, in the Riemannian spaces to be defined in GRT, the comoving world lines possess acceleration, and forces of inertia therefore arise as reaction forces of the spaces, analogous to the forces of inertia that appear in Euclidean space in Newtonian mechanics.

The preceding conclusions indicate that the approximate solutions in GRT or in relativistic theories in general, like the exact solutions in specified spaces, possess a remarkable property: the free motion of an individual material point—test particle—in a gravitational field is independent of its mass.

It should be clear that the validity of the notion "individual point" is connected with the admissible and accepted modelling in the theory proposed here. For example, in many (but of course not all) situations, the Earth, or even any star, irrespective of all its manifold internal pecularities, may be treated as a material point.

The theory developed here holds for any given family of world lines L. In that case, however, it follows from Eqs (2.14) that

$$a_\alpha dx^\alpha = -\frac{\partial U}{\partial x^\alpha} dx^\alpha = -dU(x^1, x^2, x^3) \tag{3.2}$$

Consequently, the momentum equations (2.14) may be viewed as conditions for integrability of the differential form $a_\alpha dx^\alpha$, where **a** is the absolute acceleration at the points of the family of world lines L, on which the accelerations must possess a potential $U(x^1, x^2, x^3)$. This is equivalent to assuming that there are no vortices in the family L or in the gravitational field; this follows from the axiom that the gravitational energy is a scalar, as represented by the fact that U depends only on the coordinates and figures in the basic formula (2.8) for ΛdV_4.

When formula (2.8) contains a term with U^*, which depends not only on the coordinates but on other arguments with tensor and scalar thermodynamical parameters, Eqs (2.14) and (3.2) will generally fail to hold.

In Newtonian theory and the relativistic theory of gravitation discussed here, Eqs (2.14) and (3.2), which guarantee better agreement with observations and specially set-up Newtonian experiments, should be retained. However, the main point of the more accurate modelling approach is that it incorporates the transition to observers in a four-dimensional pseudo-Euclidean Riemannian space. These relationships still allow one very considerable freedom, thanks to the choice of the potential $U(x^1, x^2, x^3)$.

Given an arbitrary potential U in a pseudo-Riemannian space, one can use Eqs (2.14) to determine the family of world lines L. It is clear, however, that the resulting mathematical solution will not always correspond to reality. For example, in Newtonian mechanics, if one chooses a function U for the gravitational field that does not satisfy Poisson's equation, the results will disagree explicitly with experiment.

An obligatory condition for determining U in Newtonian mechanics is the law of universal gravitation; allowing the absolute time parameter τ to be variable, this law may be written locally in any system of three-dimensional coordinates x^α or ξ^α as Poisson's equation

$$\Delta U = \nabla^\alpha \nabla_\alpha U = -4\pi\rho G \quad (\alpha = 1, 2, 3) \tag{3.3}$$

It is a familiar fact that any solution of Poisson's equation for U, given the dependence of the density distribution ρ in three-dimensional volumes of Euclidean space, is an equivalent exact formulation of the law of universal gravitation for the magnitude of the specific mass energy of the gravitational field. In keeping with the meaning of the definition of Riemannian spaces, the differential equation (3.3) may be introduced at each point of space in the local inertial tetrads, or in the global metric for curved Riemannian spaces.

Equation (3.3) and, accordingly, the law of universal gravitation, are valid in Newtonian mechanics in a three-dimensional global Euclidean space and have been confirmed to a very high degree of accuracy in terrestrial experiments with fixed bodies and observers of free motions of various masses in space.

Equation (3.3) may also be derived from the basic variational equation (2.8) by varying the empirical scalar c, both in Newtonian mechanics in Euclidean space and in volumes of empty space with $R_{ij} = 0$ for the relativistic models of comoving coordinate frames, when $U = U(x^1, x^2, x^3)$.

An analogous equation, as a natural correct mathematical generalization of Eq. (3.3) in relativity theory, may also be postulated for Riemannian spaces as a direct generalization of Eq. (3.3) in the comoving coordinate frame, or after variation of the constant c occurring in ΛdV_4 in the term (2.6). (The quantity c may be treated in relativistic models as a parameter that varies in the model beyond the possible experimental error in measurements of c.)

For steady fields in the observer's frame, the solution of Eq. (3.3) in his comoving variables z^1, z^2, z^3, τ', where τ' is the observer's proper time, must have the form $U(z^1, z^2, z^3)$, that is, it will not depend on τ'.

In classical GRT the term $\rho(dU/d\tau)\, dV_4$ does not occur. Hence $\mathbf{a} = \mathbf{g} = 0$ and the families of world lines L of test mass elements in our theory are special forms of a family of orbits formed by geodesic lines.

In the formulation of the problem in GRT, as we know, gravitational forces between the planets are essentially ruled out; similarly, there are no such forces between moving particles in dust clouds, provided that there are no collisions. In addition, in the approximate theory of planetary motion one ignores perturbations of the properties of planetary spaces, so that one in fact treats the motions of the planets as test masses in the specified space (in practice—along geodesics in Schwarzschild space) generated by the Sun; in particular, no allowance is made for the interactions among the planets and the effect of the oblate shape of the Sun. On the other hand, in GRT, when $U = \text{const}$ and there is still no term with U^* in (2.8), the equation in a volume of empty space V_4 with $\rho = 0$ gives

$$R_{ij} - \tfrac{1}{2}g_{ij}R = 0, \quad \text{or} \quad R_{ij} = 0 \tag{3.4}$$

These equations are not uniquely solvable. To determine the metric of the Weyl tensor (which is equal to the Riemann tensor), therefore, one should pick out a specific Weyl space and corresponding families of world lines L, permitting the absolute accelerations $\mathbf{a}$ to differ from zero. This may be done by incorporating the additional conditions that must be imposed on invariants in the case of partial differential equations with more than one solution.

It is also easy to deduce from our conclusions that all the tensor equations in the problem settings that we have described in Riemannian spaces and in Newtonian mechanics in the comoving coordinate frames are the same when $\mathbf{v} = 0$.

In particular, there is a universal relation between the absolute acceleration vectors at points of world lines with identical coordinates x^α in GRT (Minkowski space) and in Newtonian mechanics (Euclidean three-dimensional space)

$$\mathbf{a}_{\text{Min}} = \frac{d\mathbf{v}}{d\tau}\frac{1}{1 - v^2/c^2} + \frac{v(\mathbf{v}d\mathbf{v}/d\tau)}{c^2(1 - v^2/c^2)^2} \tag{3.5}$$

At each point of both spaces, with their different metrics, the three-dimensional velocities are nevertheless identical: $\mathbf{v}_{\text{Min}} = \mathbf{v}_{\text{New}} = \mathbf{v}$.

It follows from (3.5) that in the comoving reference frames, for which $\mathbf{v} = 0$ at each point

$$\mathbf{a}_{\text{Min}} = \mathbf{a}_{\text{New}} \tag{3.6}$$

Hence it follows that in the comoving coordinate frames the gravitational fields of the absolute accelerations of the forces of gravity in Euclidean three-dimensional spaces in Newtonian theory and in the corresponding three-dimensional subspaces in Minkowski pseudo-Riemannian four-dimensional space are the same, and should be determined by the same formulae in accordance with Eq. (3.3), in terms of the density $\rho(\xi^1, \xi^2, \xi^3)$ or $\rho(x^1, x^2, x^3)$.

The possible differences between laws of motion in relativistic and Newtonian mechanics are due to the different metrics, which affect the transformations from the comoving reference frame to the observer's frame via the rescaling algorithms of the theory of inertial navigation in its Newtonian versus its relativistic versions.

It should also be noted that the scalar function U of the coordinates actually affects the state of motion of point elements in the medium (which has the dimension of velocity squared) only for particles (or elements of a continuous medium) that possess mass or momentum, via energy and gravitational force.

4. It should be clear that the theory we have developed—and its implications—may be generalized to the application of model elements of continuous media with zero masses on lines L for which $dm = 0$ but the momenta do not vanish. For example, for neutral model objects with no mass such as photons or neutrinos.

Such model objects may be introduced into the theories irrespective of the subsequent physical complications for previously introduced model particles of the same types.

In fact, questions of the choice of the possible fundamental spaces and time in empty space may be considered in isolation from the concepts of the intrinsic characteristics of the elements of model media, which are embedded in the geometrical spaces under consideration as mathematical sets of points endowed with special properties.

Putting $U^* = 0$ in (2.12) and cancelling out the arbitrary mass $dm \neq 0$, we see that the expression in brackets vanishes.

We will now consider the construction of relativistic models of a continuous medium embedded in a prescribed fundamental Riemannian space whose metric is defined by some Weyl tensor, when all the infinitesimal elements of the medium have masses $dm = 0$ but their characteristic momenta are $\mathbf{p} \neq 0$, say $\mathbf{p} = k_1 \mathbf{c}$, and their energy is $\epsilon_\gamma = qc^2$, where k_1 and q are given scalar factors.

In the effective formulae (2.12) and (2.13) one can replace the velocity $\mathbf{u}$ and $\rho dU/d\tau$ by $\mathbf{p}(p^i)$ and U_ϵ with non-zero world lines L as envelopes of the vectors $\mathbf{p}$; Eqs (2.14) remain formally the same

$$d\mathbf{p}/d\tau = -\partial U_\epsilon/\partial x^\alpha \text{ and } \partial U_\epsilon/\partial\tau = 0 \tag{4.1}$$

If the world lines L in four-dimensional space are zero lines, then $d\tau = 0$ on L; for three-dimensional volumes dV_3, however, one can introduce different reference frames at the points of L, with $d\lambda \neq 0$ and metric $ds^2 = c^2 d\lambda^2 - dl^2$, in which the vector dl satisfies the equality

$$d\mathbf{l}/d\lambda = \mathbf{c} \text{ and } |\mathbf{c}| = \text{const} \tag{4.2}$$

In other words, the points of lines L in any reference frame in three-dimensional space in a vacuum will correspond to velocities (3.5) of magnitude equal to $\mathbf{c}$; and for the vector $\mathbf{p}$ we obtain

$$\mathbf{p} = \sqrt{\epsilon/q}\,\mathbf{c} = k_1 \mathbf{c} \tag{4.3}$$

The deflections of rays of light in the gravitational field will depend on the constant $\sqrt{\epsilon/q} = k_1$.

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